JOURNAL OF COMPUTATIONAL PHYSICS 53, 124–151 (1984)

Hyperbolic Two-Pressure Models for Two-Phase Flow*

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Received September 1, 1982; revised June 21, 1983

For some time it has been known that many of the two-phase flow models lead to ill-posed Cauchy problems because they have complex characteristic values. A necessary condition (at least in the linear case) for the Cauchy problem to be well-posed is that it be stable in the sense of von Neumann. For systems of partial differential equations of first order, stability in the sense of von Neumann is essentially equivalent to the condition that the model be hyperbolic (all real characteristic values and complete set of characteristic vectors). Herein models are developed which have real characteristic values for all physically acceptable states (state space) and except for a set of measure zero have a complete set of characteristic vectors in state space. Therefore, these models are hyperbolic a.e. (almost everywhere) in state space. Also, they are stable in the sense of von Neumann a.e. in state space even without inclusion of viscosity terms. The models discussed herein are developed for the case of two-phase separated planar flow and include transverse momentum considerations. These models are referred to as "two-pressure" models because each phase is assumed to exist at an average pressure different from the average pressure in the other phase; the pressure fields are related through momentum considerations. Numerical results on a steady-state problem show good agreement with existing steady-state results. Numerical results on a transient problem agree with a single-pressure model until the onset of numerical instability in the single-pressure model. Compared to the single-pressure (hydrostatic) model, the two-pressure model approximates additional physical features and is shown to be a viable approach for the case of separated flow.

1. INTRODUCTION

Development of the two-pressure models was motivated by the fact that the existing single-pressure, one-dimensional models for the flow of a two-phase fluid do not result in a system of hyperbolic equations because they have complex charac-

* Work supported by the U. S. Regulatory Commission, Office of Reactor Safety Research under DOE Contract DOE-AC07-76ID01570.

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0021-9991/84 $3.00

teristic values (or eigenvalues) [1]. Let p_1 and p_2 be the pressures in the two-phases (or materials). The complex characteristic values of the single pressure models appear to result from the unrealistic assumption (called the "hydrostatic assumption" [2]) that the p_1 and p_2 pressures are in equilibrium (this would require unrealistic signal propagation speeds in a dynamic flow situation). Whereas, the two-pressure models allow for the possibility that the flow is not hydrostatic and thus do not include the unrealistic assumption which apparently leads to the complex characteristic values.

The multidimensional balance laws of continuum mechanics apply at each spatial point which at any given time is occupied by only one fluid phase. The interfaces between the two phases are moving boundaries across which appropriate interior boundary conditions apply. Richtmyer and Morton [3, pp. 352–383] give a very good discussion of the interface instabilities of Rayleigh–Taylor and Kelvin–Helmholtz and how these instabilities affect the well-posedness of the multidimensional problem.

The one-dimensional single-pressure models for two-phase flow are obtained by an averaging process through which the multidimensional case is reduced to one spatial dimension [4]. As mentioned above, the single-pressure models have complex characteristic values within the range of interest of the dependent variables. Thus, the models are physically unacceptable and lead to ill-posed initial value problems.

A careful inspection [5] of all of the approximations and assumptions in the development of the single-pressure model led to the conclusion that the only reasonable change leading to a hyperbolic model is the change in the "hydrostatic assumption" to allow the model to become a "truly two-pressure model." In the next two paragraphs we explain what we mean by a "truly two-pressure model."

Apparently Long [2] in 1956 was the first to introduce what might be called a "two-pressure" model. However, his "two-pressure" model was based on the "hydrostatic assumption." When the gravitational field vanishes, the "hydrostatic assumption" reduces to the "equal-pressures assumption," i.e., a single-pressure model. Other "two-pressure" models, which also reduce to the single-pressure models when certain parameters vanish, are the models of Ramshaw and Trapp [6] in 1978 (where the difference in pressure was proportional to the surface tension) and of Bedford and Drumheller [7] in 1978 (where the difference in pressure was proportional to bubble inertia).

It appears that Ransom and Scofield [8] in 1976 were the first to document a two-pressure model that was hyperbolic and remained a two-pressure model even when the gravitational field, surface tension, bubble inertia, viscous stress, etc., vanished. This was the eight-equation, two-pressure model presented herein. The five-equation, two-pressure model (essentially a special case of the eight-equation model) presented herein was discovered independently by Hicks [5] in 1981. Recently, Harlow and Wendroff [9] independently discovered the eight-equation model, and after becoming aware of the earlier work encouraged the authors to prepare this paper. Further introductory and background material may be found in References [10–21].

It appears that some discussions about the single-pressure model have been obscured by some confusion about "physical instabilities" and "unphysical instabilities" and their relation to well-posed or ill-posed problems. Briefly and roughly put: unphysical (unbounded) instabilities in the model lead to ill-posed problems, whereas physical (bounded) instabilities in the model do not lead to ill-posed problems. This is discussed with more detail and precision in Section 3. The instability exhibited by the single-pressure model (in the absence of surface tension or dissipative mechanisms such as viscous stresses, thermal conduction, etc.) is an unbounded instability in which the amplification factor $A(k)$ for the kth wave number goes to infinity as $k \to \infty$ (see Sect. 3). It seems reasonable to call this unbounded instability an unphysical instability. The actual physical instabilities of interfaces that we observe in nature (e.g., wind over water, heavier fluid over lighter fluid) are bounded instabilities. Moreover, it seems that these physical instabilities usually have the characteristic that $A(k)$ attains a finite maximum greater than unity at some finite value $k = k_{\max}$ ($k_{\max}$ depends on the coefficients of surface tension, viscous stresses, etc.) and as $k \to \infty$, $A(k)$ goes to some limit less than or equal to unity (see [3, p. 358]). However, to be stable in the sense of von Neumann essentially requires only that $A(k)$ be bounded uniformly in k (see Sect. 3 for a more precise statement). Examples of mathematical models of these types of physical instability are the interface instabilities of Kelvin–Helmholtz and Rayleigh–Taylor when sufficient physics (e.g., surface tension, etc.) are included in the model (see [3]). On the other hand, when sufficient physics are not included in the model then these interface instabilities can become unphysical (unbounded) instabilities (see [3] and also see Landau and Lifshitz [22, pp. 114–116]). For further background material on the Kelvin–Helmholtz and Rayleigh–Taylor instabilities see Birkhoff [23].

From the observations in the previous paragraph it is perhaps not too surprising that the single-pressure model exhibits unbounded instabilities in the absence of surface tension, viscous stresses, etc. Indeed, it may even seem surprising that we can generate a stable model at all without the inclusion of surface tension or viscous stresses. In response to this view, we submit that the averaging process used in developing the two-pressure model probably contributes to its stability. On the other hand, essentially the same averaging process is used in the single-pressure model and it is unstable. The resolution of this seeming paradox may be that the equilibrium-pressure approximation ("hydrostatic assumption") is so unstabilizing that it overwhelms the stabilizing influence of the averaging process.

Wendroff [9, 15] has pointed out that the observations made in the previous paragraphs might lead one to the following conjecture: "In deriving the two-pressure model one assigns an interface velocity, implying that the velocity is *continuous* at the interface. In the one pressure model the velocity is discontinuous at the interface. This is probably why ill-posedness results in the latter: the problem being modeled is the ill-posed tangential discontinuity." Thus we have two plausible conjectures about what in the derivation causes the complex characteristics values in the single-pressure models:

(1) Without sufficient physics (e.g., surface tension) in the multidimensional model the tangential discontinuity instability becomes an unphysical instability, and

starting the derivation from an unphysically unstable model leads to the complex characteristic values in the single-pressure model.

(2) Agreed, that the multidimensional model without sufficient physics is unphysically unstable. Even so, if the modeling is done carefully so that no overwhelmingly unstabilizing approximations are made then there is a chance (proved possible by the two-pressure model) that the averaging process will stabilize the final model. Unfortunately, it appears that an overwhelmingly unstablizing approximation *was* made, namely, the "hydrostatic assumption": assuming equilibrium pressures in a dynamic flow model is tantamount to assuming unrealistic (infinite) signal propagation speeds and this leads to the unrealistic (complex) characteristic values of the single-pressure models.

We lean toward the latter conjecture but admit the plausibility of the former.

In summary, leaving all conjectures aside, what we can say rigorously is that the two-pressures model is stable in the sense of von Neumann a.e. in state space. On the other hand, the single-pressure model does not even have real characteristic values. For these, and other reasons to be discussed, it seems to us that the two-pressure model is the better model.

2. Model Development

The models are derived from the hydrodynamic equations for plane flow in the spatial reference frame. Let the differential operator D be defined as follows:

$$Df = \partial f/\partial t + \partial uf/\partial x + \partial vf/\partial y, \tag{2.1}$$

where f is an arbitrary function of (x, y, t), t is the temporal coordinate, (x, y) are the spatial coordinates and (u, v) is the fluid velocity vector. Then the equations may be written in vector form as follows.

$$D\mathbf{f} + \partial \mathbf{F}/\partial x + \partial \mathbf{G}/\partial y = \mathbf{S}, \tag{2.2}$$

where

$$\mathbf{f} = (\rho, \rho u, \rho v, \rho E)^{\mathrm{T}} \tag{2.3}$$

and

$$\mathbf{F} = (0, p, 0, up)^{\mathrm{T}}, \tag{2.4}$$

$$\mathbf{G} = (0, 0, p, vp)^{\mathrm{T}}, \tag{2.5}$$

and

$$\mathbf{S} = (S^{\rho}, S^{\rho u}, S^{\rho v}, S^{\rho E})^{\mathrm{T}}. \tag{2.6}$$

Here ρ is the mass density, E is the specific (total) energy,

$$E = e + (u^2 + v^2)/2.$$

The specific internal energy is e given by the equation of state

$$e = e(\rho, s). \tag{2.7}$$

The specific entropy is s, p is the pressure, and the components of the vector $\mathbf{S}$ are the source functions for the components of $\mathbf{f}$. From elementary thermodynamics

$$p(\rho, s) = \rho^2 \partial e(\rho, s)/\partial \rho, \tag{2.8}$$

$$T(\rho, S) = \partial e(\rho, s)/\partial s, \tag{2.9}$$

and

$$de = p\, d\rho/\rho^2 + T\, ds, \tag{2.10}$$

where T is the absolute temperature. Also well known is the fact that from Eq. (2.2) it follows that

$$D\rho e + p(\partial u/\partial x + \partial v/\partial y) = S^{\rho E} - uS^{\rho u} - vS^{\rho v} + S^{\rho}(E - e), \tag{2.11}$$

and Eqs. (2.10) and (2.11) lead to

$$D\rho s = S^{\rho s}, \tag{2.12}$$

where

$$TS^{\rho s} = (E - 2e + Ts - p/\rho)\, S^{\rho} + S^{\rho E} - uS^{\rho u} - vS^{\rho v}. \tag{2.13}$$

The reason we mention these well-known equations here is that we shall see analogs of these point equations in the averaged equations of the two-phase flow models.

For certain purposes (such as the analysis of weak or generalized solutions) the so-called conservative form (Eq. (2.2)) of the hydrodynamic equations is more convenient. But for purposes of obtaining the characteristic values of the hydrodynamic equations it is more convenient to replace the last equation of the system of equations, Eq. (2.2), namely the conservation of energy equation

$$D\rho E + \partial up/\partial x + \partial vp/\partial y = S^{\rho E},$$

with the equivalent entropy balance equation, Eq. (2.12). Equation (2.13) gives the equivalence relation between the source terms in the two different formulations.

Consider the very simple case of planar, stratified two-phase flow as illustrated in Fig. 1.

Figure 1 illustrates a planar, separated flow between plates at $y = 0$ and $y = H$ with the interface between fluid $n = 1$ and fluid $n = 2$ at $y = Y$, where Y is a function of x

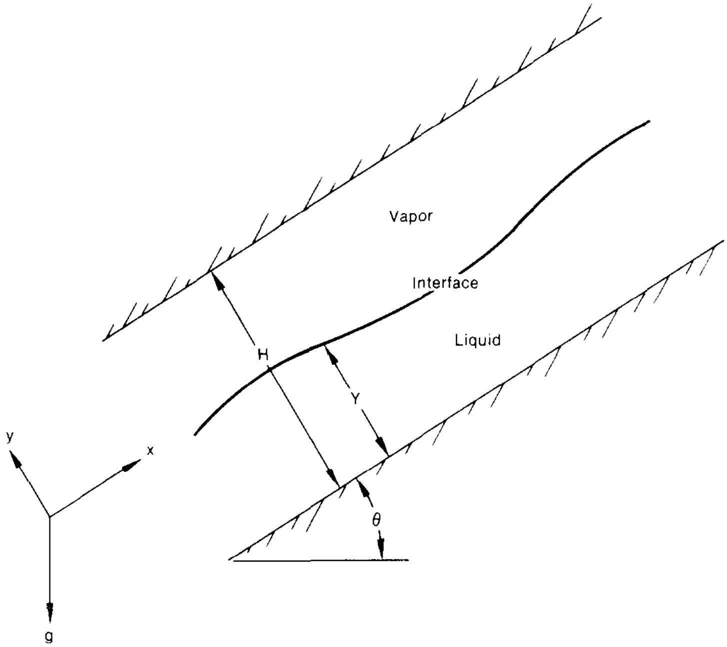

FIG. 1. Two-phase plane separated flow geometry.

and t. We have arbitrarily chosen fluid $n = 1$ to occupy the region $0 \leqslant y \leqslant Y$, and fluid $n = 2$ to occupy the region $Y \leqslant y \leqslant H$, in Fig. 1, and will follow this convention in the following derivation.

Next let α_n be the volume fraction of phase (or material) n and assume that the phases do not detach from one another nor from the plates. Then

$$\alpha_1 + \alpha_2 = 1 \tag{2.14}$$

and

$$Y = \alpha_1 H. \tag{2.15}$$

Boundary notations. Let the value of f at $y = 0$ be denoted f_0 and at $y = H$ be denoted f_H for $f = p, u, v, \rho$, etc. Let $\hat{f}_n$ be the notation for functions at the interface in phase n. That is,

$$\hat{f}_1(x, t) = f_1(x, Y-, t) \tag{2.16}$$

and

$$\hat{f}_2(x, t) = f_2(x, Y+, t). \tag{2.17}$$

Point functions. Let $f._n$ be the notation for point functions in phase n. That is,

$$\begin{aligned} f._n(x, y, t) &= f(x, y, t) \qquad \text{if point } (x, y, t) \text{ is in phase } n, \\ &= \text{zero} \qquad \text{if } (x, y, t) \text{ is not in phase } n. \end{aligned} \tag{2.18}$$

Averaging operators and the average-value functions. Let f_n be the notation for the average-value functions in phase n. That is, the f_n are defined by the following averaging operators applied to the $f._n$

$$f_n(x, t) = \int_0^H f._n(x, y, t)\, dy/\alpha_n H. \tag{2.19}$$

Denote these averaging operators by A_n. That is,

$$f_n = A_n(f._n). \tag{2.20}$$

Another notation that will be convenient defines the transverse velocity of phase n at the interface, i.e.,

$$\dot{Y}_n = \partial Y/\partial t + \hat{u}_n \partial Y/\partial x. \tag{2.21}$$

Mass transfer across the interface. For simplicity of exposition, let us first consider the case when

$$\left| \frac{\partial Y}{\partial x} \right| \ll 1. \tag{2.22}$$

In a forthcoming paper we will consider the relaxation of this constraint for more general (but more complicated) models. Let $\dot{m}$ = mass/area/sec transferred across the interface from material $n = 2$ to $n = 1$. Then by conservation of mass flowing through the interface we have

$$\dot{m} = \hat{\rho}_n [\dot{Y}_n - \hat{v}_n] \tag{2.23}$$

Equations before Averaging

For completeness of exposition augment the **f**, **F**, **G**, and **S** of Eqs. (2.3)–(2.5) to include the entropy Eq. (12). That is, define **f**., **F**., **G**., and **S**. as

$$\mathbf{f}. = (\rho., \rho.u., \rho.v., \rho.E., \rho.s.)^{\mathrm{T}}, \tag{2.24}$$

$$\mathbf{F}. = (0, p., 0, u.p., 0)^{\mathrm{T}}, \tag{2.25}$$

$$\mathbf{G}. = (0, 0, p., v.p., \mathrm{O})^{\mathrm{T}}, \tag{2.26}$$

and

$$\mathbf{S}. = (S.^{\rho}, S.^{\rho u}, S.^{\rho v}, S.^{\rho E}, S.^{\rho s})^{\mathrm{T}}. \tag{2.27}$$

As an example of source terms, those in [8] are

$$S._n^{\rho} = 0, \tag{2.28}$$

$$S._n^{\rho u} = \rho._n\, g \sin\theta - F._{nx}, \tag{2.29}$$

$$S._n^{\rho v} = \rho._n\, g \cos\theta - F._{ny},$$

$$S._n^{\rho s} = (\dot{q}/T + uF_x/T + vF_y/T)._n,$$

where g is the acceleration due to gravity, θ is the angle between the x-axis and the local horizontal, (F_x, F_y) is the vector of viscous stresses and other momentum sources, and $\dot{q}$ is the thermal energy source.

The equations before averaging may be written

$$D\mathbf{f}._n + \partial\mathbf{F}._n/\partial x + \partial\mathbf{G}._n/\partial y = \mathbf{S}._n. \tag{2.30}$$

Observe that by Leibnitz' rule

$$\partial f_n \alpha_n = A_n(\partial f._n)\,\alpha_n + \hat{f}_n \partial\alpha_n \tag{2.31}$$

for the linear differential operators $\partial = \partial/\partial t$, $\partial/\partial x$, and $\partial/\partial y$. Also note the special cases

$$\alpha_n A_n[\partial f._n/\partial x] = \partial\alpha_n f_n/\partial x - \hat{f}_n \partial\alpha_n/\partial x, \tag{2.32}$$

and letting

$$\begin{aligned} \Delta_n f_n &= \hat{f}_1 - f_0 \qquad \text{if} \quad n = 1, \\ &= f_H - \hat{f}_2 \qquad \text{if} \quad n = 2, \end{aligned} \tag{2.33}$$

yields

$$\alpha_n A_n[\partial f._n/\partial y] = \Delta_n f_n/H. \tag{2.34}$$

Next note that

$$\begin{aligned} \alpha_1 A_1[Df._1] = {} & \partial\alpha_1 f_1/\partial t + \partial\alpha_1 A_1(u._1 f._1)/\partial x + \Delta_1 v_1 f_1/H \\ & - \hat{f}_1(\partial\alpha_1/\partial t + \hat{u}_1 \partial\alpha_1/\partial x), \end{aligned} \tag{2.35}$$

and from the mass transfer equations,

$$-\dot{m}/\hat{\rho}_1 = \hat{v}_1 - \dot{Y}_1, \tag{2.36}$$

and recalling that $Y = \alpha_1 H$ leads to

$$\alpha_1 A_1(Df._1) = \partial\alpha_1 f_1/\partial t + \partial\alpha_1 A_1(u._1 f._1)/\partial x - (f_0 v_0 + \hat{f}_1 \dot{m}/\hat{\rho}_1)/H, \tag{2.37}$$

and similarly

$$\alpha_2 A_2(Df\cdot_2) = \partial\alpha_2 f_2/\partial t + \partial\alpha_2 A_2(u\cdot_2 f\cdot_2)/\partial x + (f_H v_H + \hat{f}_2 \dot{m}/\hat{\rho}_2)/H. \quad (2.38)$$

The equations above may be summarized concisely by letting

$$D_n(\alpha_n f_n) = \partial\alpha_n f_n/\partial t + \partial\alpha_n A_n(u\cdot_n f\cdot_n))/\partial x, \quad (2.39)$$

$$B_1(f) = -(f_0 v_0 + \hat{f}_1 \dot{m}/\hat{\rho}_1)/H, \quad (2.40)$$

and

$$B_2(f) = +(f_H v_H + \hat{f}_2 \dot{m}/\hat{\rho}_2)/H, \quad (2.41)$$

then Eqs. (2.37) and (2.38) are expressed as

$$\alpha_n A_n(Df\cdot_n) = D_n(\alpha_n f_n) + B_n(f_n). \quad (2.42)$$

Averaging the point-function Eq. (2.30) and multiplying by α_n yields

$$\alpha_n A_n(\mathbf{Df}\cdot_n) + \alpha_n A_n(\partial\mathbf{F}\cdot_n/\partial x) + \alpha_n A_n(\partial\mathbf{G}\cdot_n/\partial y) = \alpha_n \mathbf{S}_n. \quad (2.43)$$

Then using Eqs. (2.42), (2.34), (2.32), and (2.31) leads to

$$D_n(\alpha_n \mathbf{f}_n) + \partial\alpha_n \mathbf{F}_n/\partial x - \hat{\mathbf{F}}_n \partial\alpha_n/\partial x + \Delta_n \mathbf{G}_n/H = \alpha_n \mathbf{S}_n - B_n(\mathbf{f}). \quad (2.44)$$

Making the approximation that the average of the product is equal to the product of the averages (this approximation was investigated in [5] and it does not appear to contribute significantly to the instability nor stability of the resulting equations) and writing out the components of Eq. (2.44) in detail leads to the following averaged equations:

The averaged mass balance equations are

$$\partial\alpha_n \rho_n/\partial t + \partial\alpha_n u_n \rho_n/\partial x = \alpha_n S_n^{\rho} - B_n(\rho), \quad (2.45)$$

where

$$B_1(\rho) = -(\rho_0 v_0 + \dot{m})/H$$

and

$$B_2(\rho) = +(\rho_H v_H + \dot{m})/H.$$

The averaged axial momentum balance equations are

$$\partial\alpha_n \rho_n u_n/\partial t + \partial\alpha_n \rho_n u_n u_n/\partial x + \partial\alpha_n p_n/\partial x - \hat{p}_n \partial\alpha_n/\partial x = \alpha_n S_n^{\rho u} - B_n(\rho u), \quad (2.46)$$

where

$$B_1(\rho u) = -(v_0 \rho_0 u_0 + \hat{u}_1 \dot{m})/H$$

and

$$B_2(\rho u) = +(v_H \rho_H u_H + \hat{u}_2 \dot{m})/H.$$

The averaged transverse momentum balance equations are

$$\partial \alpha_n \rho_n v_n/\partial t + \partial \alpha_n \rho_n u_n v_n/\partial x + \Delta_n p_n/H = \alpha_n S_n^{\rho v} - B_n(\rho v), \tag{2.47}$$

where

$$B_1(\rho v) = -(\rho_0 v_0^2 + \hat{v}_1 \dot{m})/H$$

and

$$B_2(\rho v) = +(\rho_H v_H^2 + \hat{v}_2 \dot{m})/H.$$

The averaged energy balance equations are

$$\partial \alpha_n \rho_n E_n/\partial t + \partial \alpha_n \rho_n u_n E_n/\partial x + \partial \alpha_n u_n p_n/\partial x - \hat{u}_n \hat{p}_n \partial \alpha_n/\partial x + \Delta_n v_n p_n/H$$
$$= \alpha_n S_n^{\rho E} - B_n(\rho E), \tag{2.48}$$

where

$$B_1(\rho E) = -(\rho_0 E_0 v_0 + \hat{E}_1 \dot{m})/H$$

and

$$B_2(\rho E) = +(\rho_H E_H v_H + \hat{E}_2 \dot{m})/H.$$

The averaged entropy balance equations are

$$\partial \alpha_n \rho_n s_n/\partial t + \partial \alpha_n u_n \rho_n s_n/\partial x = \alpha_n S_n^{\rho s} - B_n(\rho s), \tag{2.49}$$

where

$$B_1(\rho s) = -(\rho_0 s_0 v_0 + \hat{s}_1 \dot{m})/H$$

and

$$B_2(\rho s) = +(\rho_H s_H v_H + \hat{s}_2 \dot{m})/H.$$

Note that in analogy with Eq. (2.13),

$$\begin{aligned} T_n[\alpha_n S_n^{\rho s} - B_n(\rho s)] &= (T_n s_n + E_n - 2e_n - p_n/\rho_n)[\alpha_n S_n^{\rho} - B_n(\rho)] \\ &\quad + [\alpha_n S_n^{\rho E} - B_n(\rho E)] - u_n[\alpha_n S_n^{\rho u} - B_n(\rho u)] \\ &\quad - v_n[\alpha_n S_n^{\rho v} - B_n(\rho v)] \end{aligned} \tag{2.50}$$

is the relation between the entropy source terms and the mass, momentum, and energy source terms.

At this point it is convenient to introduce the following notation:

$$\$_n^f = \alpha_n S_n^f - B_n(f) \qquad \text{for} \quad f = \rho, \rho u, \rho v, \rho E, \rho s, \text{etc.},$$

and

$$\$^f = \$_1^f + \$_2^f.$$

Then the averaged equations (Eq. (2.44)) have the form

$$D_n(\alpha_n \mathbf{f}_n) + \partial \alpha_n \mathbf{F}_n / \partial x - \hat{\mathbf{F}}_n \partial \alpha_n / \partial x + \Delta_n \mathbf{G}_n / H = \$_n,$$

where

$$\$_n = (\$_n^\rho, \$_n^{\rho u}, \$_n^{\rho v}, \$_n^{\rho E}, \$_n^{\rho s})^{\mathrm{T}}.$$

We present two different two-pressure models here. First, we present the five-equation, two-pressure model; second, we present the eight-equation, two-pressure model. The five-equation, two-pressure model is simpler but more approximate than the eight-quation, two-pressure model. However, the essence of the characteristic (value, vector) structure and how this structure affects stability is contained in the simpler five-equation model and it is therefore easier to illustrate the analysis using it.

The 5-Equation, 2-pressure Model

For simplicity let $\dot{m}$, $\$_n^\rho$ and $\$_n^{\rho u}$ vanish. Also assume a mechanical equation of state. That is,

$$p_n = p_n(\rho_n). \tag{2.51}$$

Now Eqs. (2.45) and (2.46) give us four equations in five unknowns $(\alpha_1, \rho_1, \rho_2, u_1, u_2)$ when $\alpha_1 + \alpha_2 = 1$ eliminates α_2, and Eq. (2.51) eliminates the p_n from the list of unknown variables. Thus we need one more equation which we can get from Eqs. (2.15), (2.21), and (2.23). Use

$$\hat{v}_n = (\partial \alpha_1 / \partial t + \hat{u}_n \partial \alpha_1 / \partial x) H \tag{2.52}$$

and

$$\hat{v} = (\hat{v}_1 + \hat{v}_2)/2, \tag{2.53}$$

and add Eq. (2.52) for $n = 1$ and (2.52) for $n = 2$ to get

$$\hat{v} = (\partial \alpha_1 / \partial t + \hat{u} \partial \alpha_1 / \partial x) H, \tag{2.54}$$

where

$$\hat{u} = (\hat{u}_1 + \hat{u}_2)/2.$$

To define $\hat{u}$ in terms of the five unknowns let

$$\hat{u} = (u_1 + u_2)/2. \tag{2.55}$$

Using linear approximations to the Hugoniot-isentropes issuing from the $n = 1$ and $n = 2$ states, one arrives at the following approximate but simple solution for the Riemann problem between the states $n = 1$ and $n = 2$,

$$\hat{v} = (p_1 - p_2)/(a_1 + a_2) \tag{2.56}$$

and

$$\hat{p} = (p_1/a_1 + p_2/a_2)/(1/a_1 + 1/a_2), \tag{2.57}$$

where

$$a_n = c_n \rho_n$$

is the acoustic impedance and $c_n = c_n(\rho_n)$ is the isentropic sound speed of material n given by

$$c_n^2 = dp_n/d\rho_n.$$

In summary, the 5E2P (5-equations, 2-pressures) model is given by

$$\partial \alpha_1 \rho_1/\partial t + \partial \alpha_1 u_1 \rho_1/\partial x = 0, \tag{2.58}$$

$$\partial(1 - \alpha_1)\rho_2/\partial t + \partial(1 - \alpha_1) u_2 \rho_2/\partial x = 0, \tag{2.59}$$

$$\partial \alpha_1 \rho_1 u_1/\partial t + \partial \alpha_1 \rho_1 u_1/\partial x + \alpha_1 \partial p_1/\partial x + (p_1 - \hat{p})\,\partial \alpha_1/\partial x = 0, \tag{2.60}$$

$$\partial(1 - \alpha_1)\rho_2 u_2/\partial t + \partial(1 - \alpha_1)\rho_2 u_2 u_2/\partial x + (1 - \alpha_1)\,\partial p_2/\partial x + (p_2 - \hat{p})\,\partial(1 - \alpha_1)/\partial x = 0, \tag{2.61}$$

$$\partial \alpha_1/\partial t + \hat{u}\partial \alpha_1/\partial x = \hat{v}/H, \tag{2.62}$$

where p_n is given by Eq. (2.51), $\hat{p}$ by (2.57), $\hat{v}$ by (2.56), and $\hat{u}$ by (2.55). We shall use the 5E2P model to simplify the exposition of the analysis of the characteristics of the two-pressure models. The five characteristic values of this model are $\hat{u}$ and $u_n \pm c_n$. The 8E2P (8-equations, 2-pressures) model has these same five characteristic values plus three more real characteristic values. Further analysis of this model will be given in Section 4.

The 8E2P (8-*Equations*, 2-*Pressures*) *Model*

Here a thermomechanical equation of state

$$p_n = p_n(\rho_n, s_n) \tag{2.63}$$

is assumed. The five unknowns of 5E2P $(\alpha_1, \rho_1, \rho_2, u_1, u_2)$ are augmented to include s_1, s_2, and v, where

$$v = (\rho_1 \alpha_1 v_1 + \rho_2 \alpha_2 v_2)/(\alpha_1 \rho_1 + \alpha_2 \rho_2) \tag{2.64}$$

and v takes the place of $\hat{v}$ in the fifth equation of the 5E2P model. The first five equations of 8E2P are the same five equations as in 5E2P. Two more equations come from the entropy balance equations (2.49). The eighth equation comes from the transverse momentum balance equations (2.47) as follows: Add Eq. (247) for $n = 1$ to Eq. (2.47) for $n = 2$ to get

$$\partial\rho_m v/\partial t + \partial\rho_m u_m v/\partial x + (p_2 - p_1)/H = \$^{\rho v}, \tag{2.65}$$

where

$$\rho_m = \alpha_1\rho_1 + \alpha_2\rho_2$$

is the mixture density and

$$u_m = (\alpha_1\rho_1 u_1 + \alpha_2\rho_2 u_2)/\rho_m \tag{2.66}$$

is the x component of the mixture velocity just as v is the y component of the mixture velocity.

For simplicity of exposition we assume $\dot{m} = 0$ and then the definitions

$$\hat{p} = (p_1 + p_2)/2$$

and

$$\hat{u} = (u_1 + u_2)/2$$

will lead to conservation of energy. If $\dot{m} \neq 0$ these definitions need modification to lead to energy conservation as shall be shown. In a forthcoming report we shall consider the relaxation of the $\dot{m} = 0$ constraint.

In summary, the 8E2P model is given by the following equations:

$$\partial\alpha_1\rho_1/\partial t + \partial\alpha_1 u_1\rho_1/\partial x = \$_1^\rho, \tag{2.67}$$

$$\partial(1-\alpha_1)\rho_2/\partial t + \partial(1-\alpha_1)u_2\rho_2/\partial x = \$_2^\rho, \tag{2.68}$$

$$\partial\alpha_1\rho_1 u_1/\partial t + \partial\alpha_1\rho_1 u_1 u_1/\partial x + \alpha_1\partial p_1/\partial x + (p_1 - \hat{p})\,\partial\alpha_1/\partial x = \$_1^{\rho u}, \tag{2.69}$$

$$\begin{aligned}\partial(1-\alpha_1)\rho_2 u_2/\partial t &+ \partial(1-\alpha_1)\rho_2 u_2 u_2/\partial x + (1-\alpha_1)\,\partial p_2/\partial x \\ &+ (p_2 - \hat{p})\,\partial(1-\alpha_1)/\partial x = \$_2^{\rho u},\end{aligned} \tag{2.70}$$

$$v/H = \partial\alpha_1/\partial t + \hat{u}\partial\alpha_1/\partial x, \tag{2.71}$$

$$\partial\alpha_1\rho_1 s_1/\partial t + \partial\alpha_1\rho_1 u_1 s_1/\partial x = \$_1^{\rho s}, \tag{2.72}$$

$$\partial(1-\alpha_1)\rho_2 s_2/\partial t + \partial(1-\alpha_1)\rho_2 u_2 s_2/\partial x = \$_2^{\rho s}, \tag{2.73}$$

$$\partial\rho_m v/\partial t + \partial\rho_m u_m v/\partial x + (p_2 - p_1)/H = \$^{\rho v}. \tag{2.74}$$

The 8E2P model was first introduced in [8]. The first five characteristic values here are the same as those in the 5E2P model. There are three more characteristic values in the 8E2P model and they are u_1, u_2, and u_m. Also, the characteristic vectors associated with the first five characteristic values are the same as in model 5E2P. The last three characteristic values of model 8E2P (u_1, u_2, u_m) have characteristic vectors which are linearly independent of the first five and of each other.

Remarks about the energy and entropy equations. From

$$\rho\, de = \rho T\, ds + p\, d\rho/\rho$$

and from Eqs. (2.45) and (2.49) comes the relation

$$\begin{aligned}&\partial\alpha_n\rho_n e_n/\partial t + \partial\alpha_n\rho_n u_n e_n/\partial x + p_n[\partial\alpha_n/\partial t + \partial\alpha_n u_n/\partial x] \\ &\quad = T_n \$_n^{\rho s} + \$_n^{\rho}(e_n + p_n/\rho_n - T_n s_n),\end{aligned} \tag{2.75}$$

and therefore Eq. (2.49) is equivalent to Eq. (2.75). From Eq. (2.46) it follows that

$$\begin{aligned}&(\partial\alpha_n\rho_n u_n^2/\partial t + \partial\alpha_n\rho_n u_n^3/\partial x)/2 + \alpha_n u_n \partial p_n/\partial x + u_n(p_n - \hat{p})\,\partial\alpha_n/\partial x \\ &\quad = u_n \$_n^{\rho u} - \$_n^{\rho} u_n^2/2.\end{aligned} \tag{2.76}$$

And from Eq. (2.47) comes

$$(\partial\alpha_n\rho_n v_n^2/\partial t + \partial\alpha_n\rho_n u_n v_n^2/\partial x)/2 + v_n \Delta_n\, p_n/H = v_n \$_n^{\rho v} - \$_n^{\rho} v_n^2/2. \tag{2.77}$$

The specific (total) energy is given by

$$E_n = e_n + (u_n^2 + v_n^2)/2,$$

and therefore adding Eqs. (2.75), (2.76), and (2.77) yields

$$\begin{aligned}&\partial\alpha_n\rho_n E_n/\partial t + \partial\alpha_n\rho_n u_n E_n/\partial x + p_n[\partial\alpha_n/\partial t + \partial\alpha_n u_n/\partial x] + u_n \partial\alpha_n\, p_n/\partial x \\ &\quad - u_n\, \hat{p}\,\partial\alpha_n/\partial x + v_n \Delta_n\, p_n/H = T_n \$_n^{\rho s} + u_n \$_n^{\rho u} + v_n \$_n^{\rho v} \\ &\quad + \$_n^{\rho}[2e_n - E_n + p_n/\rho_n - T_n s_n],\end{aligned} \tag{2.78}$$

and thus Eq. (2.75) is equivalent to Eq. (2.78).

The total energy density is given by

$$\alpha_1\rho_1 E_1 + \alpha_2\rho_2 E_2$$

and adding Eq. (2.78) for $n = 1$ and Eq. (2.78) for $n = 2$ yields the balance equation for the total energy density. Using the relations for v in model 8E2P, observe that the total energy balance equation can attain conservative form when $\dot{m} = 0$ provided that

$$u_1\, p_1 - u_2\, p_2 + (u_2 - u_1)\,\hat{p} + \hat{u}(p_2 - p_1) = 0. \tag{2.79}$$

Eq. (2.79) is satisfied for example when

$$\hat{p} = (p_1 + p_2)/2 \tag{2.80}$$

and

$$\hat{u} = (u_1 + u_2)/2. \tag{2.81}$$

If $\dot{m} \neq 0$ then the RHS of Eq. (2.79) is no longer zero. The RHS for $\dot{m} \neq 0$ may be determined by using Eq. (2.78) and recalling that

$$\$_n^f = \alpha_n S_n^f - B_n(f)$$

and that the $\dot{m}$ terms are included in the $B_n(f)$ terms as shown in Eqs. (2.40) and (2.41).

Thus, when $\dot{m} \neq 0$ then Eq. (2.79) becomes

$$u_1 p_1 - u_2 p_2 + (u_2 - u_1)\hat{p} + \hat{u}(p_2 + p_1) = \sum_{n=1}^{2} [T_n B_n(\rho s) + u_n B_n(\rho u) + v_n B_n(\rho v) + B_n(\rho)[2e_n - E_n + p_n/\rho_n - T_n s_n]]. \tag{2.82}$$

Note that to conserve energy Eqs. (2.80) and (2.81) for $\hat{p}$ and $\hat{u}$ must be appropriately modified when $\dot{m} \neq 0$ so that Eq. (2.82) is satisfied.

Another interesting observation about model 8E2P is that when the $\mathbf{S}_n$ source terms and $\dot{m}$ are both zero then the entropy is conserved.

3. Background, Stability, Well-posedness

Let $\mathbf{U}$ be a vector of M components and consider a system of partial differential equations of the form

$$\partial \mathbf{U}/\partial t = \mathbf{F}(t, x, \mathbf{U}, \partial \mathbf{U}/\partial x, \ldots, \partial^n \mathbf{U}/\partial x^n) \tag{3.1}$$

whose solution is desired for the finite time interval $[t^0, t^f]$. Let the equation of first variation of Eq. (3.1) be written

$$\partial\, \delta\mathbf{U}/\partial t = \mathbf{A}(\partial/\partial x)\, \delta\mathbf{U}, \tag{3.2}$$

where δ is the variation operator and $\mathbf{A}(\partial/\partial x)$ is an M by M matrix polynomial in $\partial/\partial x$ with coefficients depending on t, x, $\mathbf{U}$, $\partial\mathbf{U}/\partial x, \ldots, \partial^n\mathbf{U}/\partial x^n$. Let $\mathbf{A}_0(\partial/\partial x)$ be $\mathbf{A}(\partial/\partial x)$ evaluated at D_0 where $D_0 = (t_0, x_0, \mathbf{U}_0, \partial\mathbf{U}_0/\partial x, \ldots, \partial^n\mathbf{U}_0/\partial x^n)$ is an $(n+3)$-tuple with constant components. Note that $\mathbf{A}_0(\partial/\partial x)$ is a linear operator with constant coefficients. The equation

$$\partial\, \delta\mathbf{U}/\partial t = \mathbf{A}_0(\partial/\partial x)\, \delta\mathbf{U} \tag{3.3}$$

is called the local (at D_0) equation of first variation of Eq. (3.1). Since $\mathbf{A}_0$ is linear with constant coefficients the related Cauchy problem is amenable to Fourier analysis. Let S_k^t be the kth Fourier component of the solution operator of the Cauchy problem for Eq. (3.3). That is,

$$S_k^t = \exp(\mathbf{A}_0(ik)\, t),$$

where k is the wave number $-\infty < k < \infty$. The von Neumann stability analysis involves the inspection of the spectral radius r_k^t of S_k^t.

DEFINITION. If r_k^t is bounded uniformly in k for all t in the finite time interval $[t^0, t^f]$, then Eq. (3.1) is said to be *stable in the sense of von Neumann* at D_0.

Observe that a model may be stable in the sense of von Neumann while being unstable in the sense of Kelvin–Helmholtz or Rayleigh–Taylor.

Consider the case when Eq. (3.1) has the form

$$\partial \mathbf{U}/\partial t = \mathbf{C}(\mathbf{U})\, \partial \mathbf{U}/\partial x + \mathbf{B}(\mathbf{U}), \tag{3.4}$$

where $\mathbf{C}(\mathbf{U})$ is a real $M \times M$ array and $\mathbf{B}(\mathbf{U})$ is a real M vector. We follow Courant and Hilbert [19, p. 425] in the following definition of hyperbolic.

DEFINITION. The matrix $\mathbf{C}$ is said to be hyperbolic at $\mathbf{U}$ if and only if $\mathbf{C}(\mathbf{U})$ has a set of characteristic values with elements all real and a set of characteristic vectors that is complete.

THEOREM 1. *Equation* (3.4) *is stable at* D_0 *in the sense of von Neumann for all* $\mathbf{B}$ *if and only if* $\mathbf{C}(\mathbf{U}_0)$ *is hyperbollic.*

Proof may be found in [5].

The four equation, one-pressure model (also called the equal-pressure or single-pressure model which, for short, we refer to as the 4E1P model) is given by the following four equations (where the source terms have been dropped for simplicity):

$$\partial \alpha_1 \rho_1/\partial t + \partial \alpha_1 \rho_1 u_1/\partial x = 0,$$

$$\partial (1-\alpha_1)\rho_2/\partial t + \partial (1-\alpha_1)\rho_2 u_2/\partial x = 0,$$

$$\partial \alpha_1 \rho_1 u_1/\partial t + \partial \alpha_1 \rho_1 u_1 u_1/\partial x + \alpha_1 \partial p/\partial x = 0,$$

$$\partial (1-\alpha_1)\rho_2 u_2/\partial t + \partial (1-\alpha_1)\rho_2 u_2 u_2/\partial x + (1-\alpha_1)\,\partial p/\partial x = 0,$$

where

$$dp = \rho_m c_m^2 [\alpha_1\, d\rho_1/\rho_1 + \alpha_2\, d\rho_2/\rho_2]$$

and

$$\rho_m c_m^2 = \rho_1 c_1^2 \rho_2 c_2^2/(\rho_1 c_1^2 + \rho_2 c_2^2).$$

For derivation of the equations above see [20].

These equations come about by the equal-pressures assumption, that is, it is assumed that

$$p_1 = p_2 = \hat{p}_1 = \hat{p}_2.$$

The six equation, one-pressure model (6E1P) augments 4E1P by including the two entropy balance equations which are needed for thermomechanical equations of state.

The characteristic polynomial for the single-pressure model can be shown (see [20]) to be

$$P_4(\lambda) = (\lambda + u_1)^2(\lambda + u_2)^2 - K_1(\lambda + u_1)^2 - K_2(\lambda + u_2)^2,$$

where

$$\rho_n K_n = \rho_m c_m^2.$$

P_4 can be simplified by the Galilean transformation $u_2 \to w$, $u_1 \to -w$, where $w = (u_2 - u_1)/2$. Then P_4 reduces to

$$P_4(\lambda) = (\lambda^2 - w^2)^2 - K_2(\lambda + w)^2 - K_1(\lambda - w)^2.$$

Let

$$\Sigma = (K_1 + K_2)/2,$$

$$\Delta = (K_1 - K_2)/2,$$

$$a_0 = w^2(w^2 - w\Sigma),$$

$$a_1 = -4w\Delta,$$

and

$$a_2 = -2(w^2 + \Sigma).$$

Then P_4 may be written as

$$P_4(\lambda) = \lambda^4 + a_2\lambda^2 + a_1\lambda + a_0.$$

The following lemma (which we call the W-lemma) turns out to be very useful in determining when the roots of the quartics that arise in the stability analyses of two-phase flow are real.

LEMMA (W). *Consider the real quartic*

$$Q(\lambda) = \lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0.$$

Necessary and sufficient conditions for Q to have four real roots are that Q' have three real roots

$$r_L \leqslant r_M \leqslant r_R,$$

such that the following three inequalities are satisfied

$$Q(r_L) \leqslant 0, \qquad Q(r_M) \geqslant 0, \qquad \textit{and} \qquad Q(r_R) \leqslant 0.$$

The plan of the proof becomes evident when the graph of a typical quartic is considered: roughly the graph has the shape of a "W." The proof may be found in [5].

To illustrate the use of the W-lemma consider the P_4 of the single-pressure model. Assume equal-phases. That is, assume $f_1 = f_2$ for $f = \alpha, \rho, K$, etc. Then $a_1 = 0$ and P_4 reduces to

$$P_4(\lambda) = \lambda^4 + a_2\lambda^2 + a_0$$

and it is easy to see that P_4 is symmetric about $\lambda = 0$, and therefore $r_M = 0$, and then condition $Q(r_M) \geqslant 0$ reduces to $a_0 \geqslant 0$, which becomes

$$w^2(w^2 - 2c^2) \geqslant 0.$$

This is satisfied if $w^2 = 0$, or if

$$w^2 \geqslant 2c^2,$$

but is not satisfied for

$$0 < w^2 < 2c^2.$$

Hence P_4 does not have real roots for all physically acceptable values of the fluid parameters (state space).

This analysis can be carried much further. Indeed, it has been carried to completion by Wendroff [9, 15]. To present his results let

$$\begin{aligned} a &= -(w^2 + \Sigma), \\ b &= -w\Delta, \\ 3\alpha &= \sin^{-1}(b/2\,|3/a|^{3/2}), \\ \lambda_k &= 2\,|a/3|^{1/2} \sin(\alpha + 2\pi k/3), \end{aligned}$$

and

$$\Lambda_k = a\lambda_k^2 + 3b\lambda_k + a_0,$$

for $k = 0, 1, 2$. Then it can be shown (see [20]) that

$$[\Lambda_0 \geqslant 0] \qquad \text{if and only if } [P_4(r_M) \geqslant 0],$$

and

$$[\Lambda_1 \leqslant 0 \text{ and } \Lambda_2 \leqslant 0] \qquad \text{if and only if } [P_4(r_L) \leqslant 0 \text{ and } P_4(r_R) \leqslant 0].$$

Therefore, by the W-lemma, the conditions

$$\Lambda_0 \geqslant 0, \qquad \Lambda_1 \leqslant 0, \qquad \Lambda_2 \leqslant 0$$

are necessary and sufficient for P_4 to have four real roots.

Wendroff [9, 15] has shown that the Λ_1 and Λ_2 constraints are vacuous and the Λ_0 constraint is equivalent to

$$(K_1^{1/3} + K_2^{1/3})^3 \leqslant 4w^2$$

which we shall refer to as the W-inequality.

Thus, Wendroff [9, 15] has proved

THEOREM 2 (Wendroff). *Consider the polynomial*

$$P_4(\lambda) = (\lambda^2 - w^2)^2 - K_2(\lambda + w)^2 - K_1(\lambda - w)^2,$$

where w is real and the K_n are nonnegative. Then there are two cases, where all the roots of P_4 are real,

(i) $K_1 = 0$ *or* $K_2 = 0$ *or* $w = 0$, *or*

(ii) $K_1 > 0$, $K_2 > 0$, $w \neq 0$, *and the W-inequality holds.*

In summary, there are several kinds of stability–instability: von Neumann, Kelvin–Helmholtz, Rayleigh–Taylor, etc. An instability in the sense of von Neumann is an unphysical instability and leads to ill-posed problems. With sufficient physics (e.g., surface tension, etc.) in the model the instabilities in the sense of Kelvin–Helmholtz and Rayleigh–Taylor are physical instabilitities and do not lead to ill-posed problems. The last two statements can be proved rigorously in the case of the linear initial value problem for a finite time interval. However, as pointed out in the Introduction, when the model does not include sufficient physics (e.g., surface tension, etc.) then the interface instabilities may become unphysical (unbounded) instabilities.

As is well known, well-posedness has three parts: existence (of the solution operator), uniqueness (of the solution operator), and stability (continuity of the solution operator in some topology). Of the aforementioned kinds of stability it appears that stability in the sense of von Neumann is the most closely related to well-posedness. In the linear case this can be proved rigorously. Theorem 1 shows that stability in the sense of von Neumann is essentially equivalent to hyperbolicity.

4. CHARACTERISTICS ANALYSES OF THE TWO-PRESSURE MODELS

DEFINITION. A state $(\alpha_1, \alpha_2, c_1, c_2, p_1, p_2, \rho_1, \rho_2, s_1, s_2, u_1, u_2, v_1, v_2)$ is *physically acceptable* provided that α_n, c_n, p_n, for ρ_n, s_n, u_n, v_n are real numbers such that $c_n \geqslant 0$, $\rho_n \geqslant 0$, $0 \leqslant \alpha_n \leqslant 1$ for $n = 1$ and 2. The set of all physically acceptable states shall be referred to as the *state space.*

Let

$$\mathbf{U}^T = (\alpha_2 \rho_2, \alpha_2 \rho_2 u_2, \alpha_1 \rho_1, \alpha_1 \rho_1 u_1, \alpha_1)$$

and let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ c_2^2 - u_2^2 & 2u_2 & 0 & 0 & -P_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & c_1^2 - u_1^2 & 2u_1 & P_1 \\ 0 & 0 & 0 & 0 & \hat{u} \end{bmatrix},$$

where

$$P_n = p_n - \hat{p} - c_n^2 \rho_n.$$

Then, model 5E2P may be written

$$\partial \mathbf{U}/\partial t + \mathbf{A} \partial \mathbf{U}/\partial x = 0.$$

Observe that

$$\det(\lambda \mathbf{I} - \mathbf{A}) = P_5(\lambda),$$

where

$$P_5(\lambda) = [(\lambda - u_2)^2 - c_2^2][(\lambda - u_1)^2 - c_1^2](\lambda - \hat{u}).$$

The five roots of P_5 are seen to be

$$u_n \pm c_n \quad \text{and} \quad \hat{u}.$$

Recall that

$$\hat{u} = (u_1 + u_2)/2.$$

For all the physically acceptable states of the material the u_n and c_n are real numbers. Therefore, we have proved

Result 1. Model 5E2P has all real characteristic values for all physically acceptable states. The characteristic values are $\hat{u} = (u_1 + u_2)/2$ and $u_n \pm c_n$ for $n = 1, 2$.

To ensure that model 5E2P is hyperbolic we must check to see if there is a complete set of characteristic vectors (or eigenvectors). We make the following observations about the characteristic vectors:

(1) Associated with the characteristic values $u_2 \pm c_2$ are the right characteristic vectors

$$(1, u_2 \pm c_2, 0, 0, 0)^{\mathrm{T}}.$$

(2) Associated with the characteristic values $u_1 \pm c_1$ are the right characteristic vectors

$$(0, 0, 1, u_1 \pm c_1, 0)^{\mathrm{T}}.$$

(3) The four characteristic vectors of (1) and (2) are linearly independent.

Next, we introduce notation for (4)–(6). Let

$$\gamma_n = c_n^2 - (u_n - \hat{u})^2$$

and consider the existence of a characteristic vector associated with the characteristic value $\hat{u}$.

(4) The (0, 0) case. When $\gamma_1 = 0$ and $\gamma_2 = 0$ there exists a characteristic vector associated with $\hat{u}$ that is linearly independent of those in (1) and (2) if and only if $P_1 = 0$ and $P_2 = 0$. The characteristic vector (modulo scalar multiples) is

$$(0, 0, 0, 0, 1)^{\mathrm{T}}.$$

(5) The $(0, \neq 0)$ case. When $\gamma_1 = 0$ and $\gamma_2 \neq 0$ there exists a characteristic vector associated with $\hat{u}$ that is linearly inependent of those in (1) and (2) if and only if $P_1 = 0$. The characteristic vector (modulo scalar multiples) is

$$(P_2/\gamma_2, \hat{u}P_2/\gamma_2, 0, 0, 1)^{\mathrm{T}}.$$

(6) The $(\neq 0, 0)$ case. When $\gamma_1 \neq 0$ and $\gamma_2 = 0$ there exists a characteristic vector associated with $\hat{u}$ that is linearly independent of those in (1) and (2) if and only if $P_2 = 0$. The characteristic vector (modulo scalar multiples) is

$$(0, 0, -P_1/\gamma_1, -\hat{u}P_1/\gamma_1, 1)^{\mathrm{T}}.$$

(7) The $(\neq 0, \neq 0)$ case. When $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$ there exists a characteristic vector associated with $\hat{u}$ that is linearly independent of those in (1) and (2) for all values of P_1 and P_2. The characteristic vector (modulo scalar multiples) is

$$(P_2/\gamma_2, \hat{u}P_2/\gamma_2, -P_1/\gamma_1, -\hat{u}P_1/\gamma_1, 1)^{\mathrm{T}}.$$

In summary, we have proved

Result 2. For all physically acceptable states the 5E2P model is hyperbolic if and only if one of the following cases is obtained:

case (0, 0):	$\gamma_1 = 0, \gamma_2 = 0$	and	$P_1 = 0, P_2 = 0.$
case (0, $\neq 0$):	$\gamma_1 = 0, \gamma_2 \neq 0$	and	$P_1 = 0.$
case ($\neq 0$, 0):	$\gamma_1 \neq 0, \gamma_2 = 0$	and	$P_2 = 0.$
case ($\neq 0$, $\neq 0$):	$\gamma_1 \neq 0, \gamma_2 \neq 0.$		

Observe that the 5E2P model always has real characteristic values when the states are physically acceptable but there are occasions ($\gamma_n = 0$ for $n = 1$ or 2) when the number of linearly independent characteristic vectors drops from five (complete set) to four (incomplete set). When this happens the system is not hyperbolic. We call this the "transonic difficulty" for the following reasons:

Observe that since

$$\hat{u} = (u_1 + u_2)/2,$$

hence

$$\gamma_n = c_n^2 - w^2,$$

where

$$w = (u_2 - u_1)/2.$$

If

$$|w| < \min_n c_n,$$

$$\max_n c_n < |w|,$$

then w is in the "supersonic" region and **A** is hyperbolic. Only as $|w|$ passes through c_1 or c_2 (i.e., the "transonic" regions) does **A** lose its hyperbolicity.

When the hyperbolicity of **A** is lost in this way then the stability of the model depends on how the source terms are defined (see [5]). However, from the following considerations, it does not seem that the transonic difficulty should be of very much concern.

If we put a natural, normalized (e.g., a reasonably weighted Lebesgue) measure on each of the state variables and the usual product measure on the state space, then the set of points in state space, where

$$\gamma_1 \neq 0, \qquad \gamma_2 \neq 0$$

is violated, is a set of measure zero (because it is contained in a subspace of lower dimension) for any physically acceptable definition of $\hat{u}$ (e.g., $\hat{u} = (u_1 + u_2)/2$ is a physically acceptable definition of $\hat{u}$). It then follows from Result 2 that the matrix **A** is hyperbolic a.e. (almost everywhere) in state space and then by Theorem 1 the 5E2P model is stable in the sense of von Neumann a.e. in state space. In summary we have

Result 3. The 5E2P model is hyperbolic a.e. in state space and stable in the sense of von Neumann a.e. in state space.

Consider the 8E2P model. The **A** associated with the 8E2P model is 8×8 and the upper left 5×5 subblock is identical with the **A** of the 5E2P model. The three equations (2 entropy balance and 1 transverse momentum balance) that are added to the 5E2P model to get to the 8E2P model contribute three new real characteristic values, u_1, u_2, and u_m. Each of the three new characteristic vectors associated with the three new characteristic values is independent of the vectors in the five-dimensional subspace associated with the first five unknowns $(\alpha_1, \rho_1, \rho_2, u_1, u_2)$. Also the three new characteristic vectors span the three-dimensional subspace associated with the last three unknowns (s_1, s_2, v). It follows in a pattern similar to the proof of Result 3 that we have the result stated below.

Result 4. The 8E2P model is hyperbolic a.e. in state space and stable in the sense of von Neumann a.e. in state space.

5. Numerical Results

Two simple computer programs were written to compare steady state and transient solutions of the 8E2P model with other models. The steady state program used the fourth-order Runge–Kutta method to integrate the system of ordinary differential equations obtained by setting all time derivatives to zero in the 8E2P model. The transient flow program used the Lax–Wendroff [3, 21] two-step explicit scheme to numerically integrate the system of partial differential equations associated with model 8E2P.

The results for two example cases, with comparisons to the results of other models, are presented to demonstrate the character of the 8E2P model. The example problems are a gravity dominated flow of water through quiescent air and a two-phase blowdown from a constant area pipe. The gravity dominated flow problem (sometimes referred to as the water faucet problem) demonstrates the ability of the model to approximate two-dimensional phenomena.

The two-phase blowdown problem tests the ability of the model to simulate rapid transients and illustrates the coupling between phases.

Gravity Dominated Flow. The gravity dominated flow problem is typical of a downcomer in a pressurized, water cooled nuclear reactor under ideal conditions where effects due to viscous stresses and phase changes can be considered negligible. Boundary conditions require the specification of appropriate dependent variables at each end of the column of water. At the top of the flow the velocities (u_n and v) and volume fractions (α_n with $\alpha_1 + \alpha_2 = 1$) were specified while at the bottom both pressures (p_n) were set equal to the ambient pressure.

The gravity dominated flow problem was solved by both the steady-state flow program and the transient flow program (by integrating until a steady state was approached in the latter case).

The results are compared to results obtained using the classical one-dimensional steady flow (C1DSF) model. These results are presented in Fig. 2. The conditions for the steady state and transient version of the 8E2P model were the same as in the C1DSF model with two exceptions: (a) the transverse velocity is not considered (explicitly) in the C1DSF model; (b) the pressure is assumed to be everywhere constant in the C1DSF model. Therefore, to obtain a valid comparison between C1DSF and the 8E2P models (both steady state and transient version), a boundary value (v_0 in Fig. 2) for the transverse velocity equivalent to that which is implicit in the C1DSF model should be used. The required boundary value of the transverse velocity was calculated from the slope of the flow boundary in the C1DSF model at the top of the downcomer.

The upper ($x = 0$) boundary conditions chosen were

liquid axial velocity	$u_1(0, t) = -10$ fps
vapor axial velocity	$u_2(0, t) = 0$
liquid volume fraction	$\alpha_1(0, t) = 0.5$
liquid density	$\rho_1(0, t) = 62.4\ \text{lbm/ft}^3$
vapor density	$\rho_2(0, t) = 0.09728\ \text{lbm/ft}^3$
liquid sound speed	$c_1(0, t) = 5000$ fps
vapor sound speed	$c_2 0, t) = 1000$ fps
transverse velocity	$v(0, t) = -1.61$ fps.

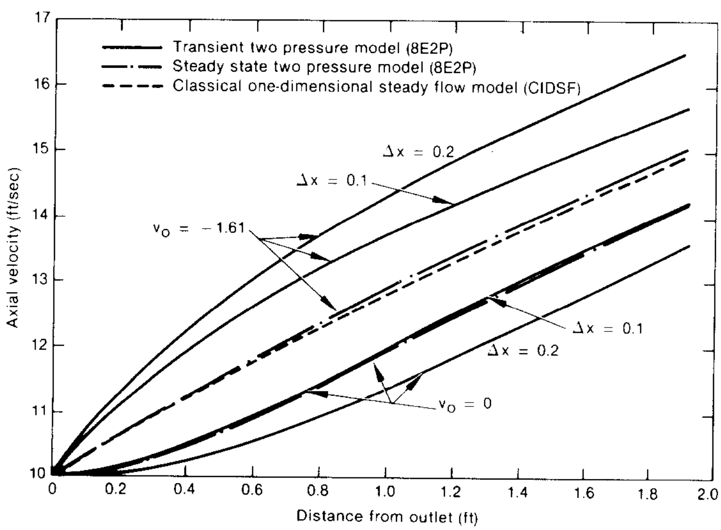

FIG. 2. Gravity dominated flow solutions.

The axial velocity (u_1) distribution calculated for each of the three models using the boundary and initial conditions described above are presented in Fig. 2. There is good agreement between the steady-state models although the velocity calculated from the steady-state 8E2P model is slightly higher than that of the C1DSF model. This is to be expected because the pressure inside the water column predicted by the 8E2P model is slightly higher than the ambient pressure assumed by the C1DSF model, thus creating an accelerating force on the liquid. The results from the steady state and transient version of the 8E2P model do not agree as well as the results for the two steady-state models, but as shown in Fig. 2 the transient solution does approach the steady-state solutions as the mesh is refined. Thus, some of the difference is due to the truncation error.

The ability of the 8E2P model to approximate two-dimensional effects is illustrated by imposing the boundary condition of parallel flow at the upper boundary ($v(0, t) = 0$). Runs were made with the 8E2P model using this boundary condition. Velocity plots from this series of runs are also shown in Fig. 2 (those labeled $v_0 = 0$).

Two-Phase Blowdown Problem. The two-phase superheated vapor and subcooled liquid blowdown problem was run using the transient versions of the 8E2P and 6E1P models. The blowdown calculations were made for a pipe 10 ft long and initially pressurized to 45 psia. Boundary conditions were: zero velocities at the closed end; constant ambient pressure at the open end. While the constant pressure boundary condition is not physically correct for all flow conditions, it was chosen for simplicity and is satisfactory for purposes of comparison. The vapor was assumed to behave as an ideal gas with specific heat ratio $\gamma = 1.4$, and a sound speed $c_2 = 1000$ fps, at the reference pressure $p_2 = 15$ psia. The liquid was assumed to have a density $\rho_1 = 62.4\ \text{lbm/ft}^3$ at $p_1 = 45$ psia, and the sound speed was assigned the constant value $c_1 = 5000$ fps.

Figure 3 plots pressure versus time in both the liquid and vapor at the closed end of the pipe for the 8E2P model. The dashed line is the vapor pressure while the solid is the liquid pressure. The overall behavior is similar to the blowdown of the superheated vapor alone. This indicates that the presence of the liquid has little overall effect on the blowdown.

There are a number of features of the solution which are unique to the 8E2P model. Oscillations in the liquid pressure at the closed end appear significantly ahead of the main decompression. This is a result of the fact that pressure disturbances in the 8E2P model are transmitted at a range of speeds rather than at a single speed as predicted by the classical single-pressure, two-phase models.

It is also evident that the differences between the two pressures are small but not insignificant. A computer generated motion picture was made showing the pressure distributions in the pipe as a function of time for this example problem. It shows clearly that the liquid's pressure distribution along the pipe exhibits a standing wave pattern that oscillates about the vapor's pressure distribution.

The closed end pressure as a function of time for the 6E1P model is shown in Fig. 4. An interesting feature of the 6E1P solution is the large pressure fluctuations

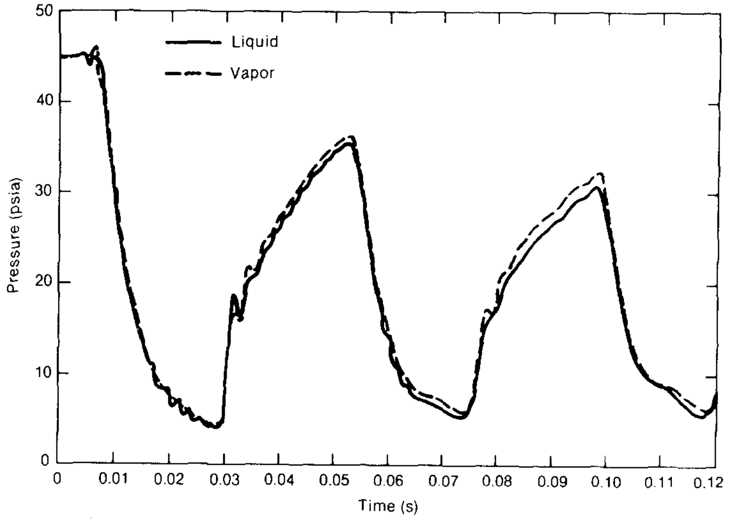

FIG. 3. Two pressure model—two phase blowdown (closed end pressures versus time).

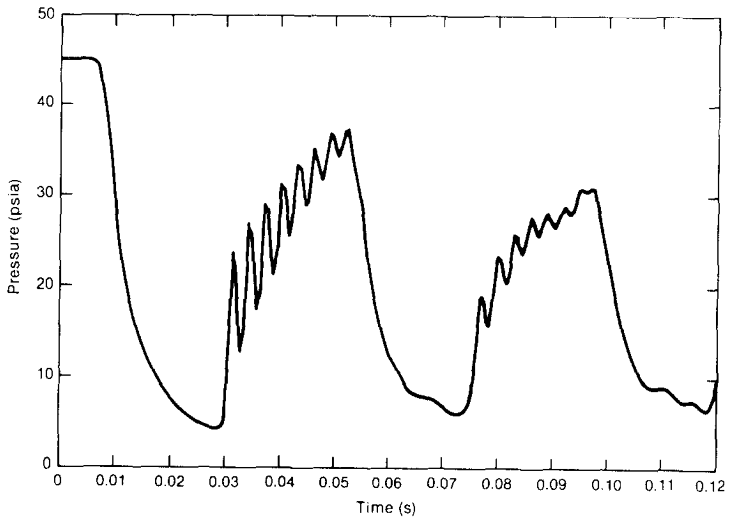

FIG. 4. Single pressure model—two phase blowdown (closed end pressure versus time).

occurring after the recompression shock has reflected from the closed end. Overlaying Figs. 3 and 4 shows good agreement between the two models when mean values are considered.

During the initial decompression (up to 30 msec) the solutions are virtually identical and they agree with results obtained for superheated vapor blowdowns.

During the blowdown, the volume fraction distributions calculated with both models developed small amplitude waves which grew slowly at first but began to grow more rapidly as the blowdown proceeded. At 120 msec a sharp interfacial wave formed near the exit and the vapor volume fraction went to zero at about 150 msec. This result is due to the predicted growth of finite wavelength disturbances (physical instability of the interface) and perhaps could be used to predict flow regime transition.

The transient solutions of both the 8E2P and 6E1P models were obtained using the explicit Lax–Wendroff [3, 21] two-step numerical integration scheme. It was expected that this procedure would result in numerical instability for the 6E1P model as the mesh is refined since this model has unreal characteristics values. Indeed, as the mesh was refined the 6E1P calculation would run a fixed number of time steps before the excursions of the volume fractions exceeded the bounds of physical acceptability, namely zero and unity, so that reasonable results could not be obtained. The 8E2P model did not exhibit this behavior and mesh refinement produced correspondingly smaller changes in the solution.

6. Concluding Remarks

This study shows that one-dimensional, transient, hydrodynamic models for two-phase separated flows can be developed which have all real characteristic values everywhere in state space, a complete set of characteristic vectors a.e. (in state space), and are stable in the sense of von Neumann a.e.

Also, these two-pressure models contain the essential physics of the two-dimensional, transient, two-phase flow. Implicit in these results is the implication that the unreal characteristic values of the single-pressure models result from the hydrostatic (or equal-pressures) assumption.

Particular conclusions are numbered as follows:

(1) The two-pressure models are hyperbolic a.e.

(2) Standard explicit, implicit, or method of characteristic integration schemes can be used to obtain numerical solutions of the two-pressure models and it is not necessary to use excessive artificial viscosity or other numerical dissipation devices such as the single-pressure models use to suppress unbounded growth due to their unphysical instabilities.

(3) The two-pressure models provide greater physical detail than the single-pressure models.

(4) The two-pressure models can simulate growth related to physical instabilities yet avoid the unphysical instabilities of the single-pressure models which obscure the true physical behavior.

ACKNOWLEDGMENTS

First, we wish to acknowledge the advice and encouragement of B. Wendroff and F. Harlow of Los Alamos who independently discovered the 8E2P model and gave us a great number of excellent suggestions on the presentation of this material. Second, we wish to thank M. P. Scofield for the work he did in programming and running the codes which provided the numerical examples. Third, we wish to thank the following people at Idaho National Engineering Laboratory who critically reviewed this paper for us, B. Vaishnavi, E. Marwil, and R. Pate.

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